On images, preimages and different notions related to APN mappings

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Notation and basic notions

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Definition

Function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ is called *almost perfect nonlinear* (APN), if $\delta_F = 2$, i.e. if equation $F(x) + F(x+a) = b$ has at most 2 solutions, $\forall a, b \in \mathbb{F}_{2^n}$, $a \neq 0$.

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$F(x_1)+F(x_3) \neq F(x_2)+F(x_4) \quad \Leftrightarrow \quad F(x_1)+F(x_1+a) \neq F(x_2)+F(x_2+a) \quad \Leftrightarrow \quad F(x)+F(x+a)$ has at most 2 solutions

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Flats and their images

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Fact: The number of vanishing flats of F can be used as a measure of how close F is to an APN function. [5]

Two generalizations of almost perfect nonlinearity

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k-strongly non-normal [2]

F is called *k-strongly non-normal* if, for every k-flat $L \subseteq \mathbb{F}_{2^n}$, the restriction $F|_L : L \to \mathbb{F}_{2^m}$ is not affine.

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Fact: An APN function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ is k-strongly non-normal for every $2 \leq k \leq n$. [2]

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Theorem

Let $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be an APN function, and let $\mathcal{P} := \{F^{-1}(a) \mid a \in \mathbb{F}_{2^n}, |F^{-1}(a)| \geq 3\}$. If $\mathcal{P} \neq \emptyset$, then the number of unbroken 2-dimensional flats is

$$|\mathcal{UF}_{n,F}| = \sum_{P \in \mathcal{P}} \binom{|P|}{3}.$$

Otherwise, $|\mathcal{UF}_{n,F}| = 0$. In particular, if F is almost k-to-1, with $k \geq 3$, then

$$|\mathcal{UF}_{n,F}| = \frac{\binom{k}{3}(2^n - 1)}{k} = \frac{(2^n - 1)(k-1)(k-2)}{6}.$$

Proof sketch :)

Setup and definition: Let $F : \mathbb{F}_{2n} \to \mathbb{F}_{2n}$ be an APN function. Let $L = \{x_1, x_2, x_3, x_4\}$ be a 2-dimensional flat.
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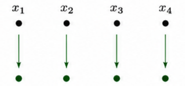
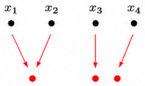


The four images are all distinct.

→ L is broken (APN).

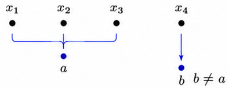
Case B: $|F(L)| = 3$



Exactly three distinct images.

→ L is broken.

Case C: $|F(L)| = 2$ (almost 3-to-one)



Three points map to a, one point to $b \neq a$.

→ L is unbroken
(maps to a 1-dimensional flat $\{a, b\}$).

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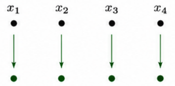

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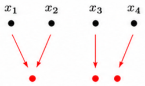

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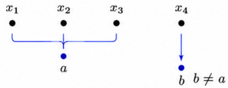

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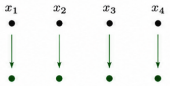

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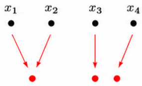

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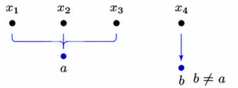

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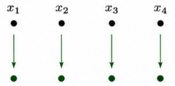

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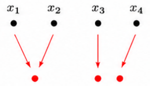

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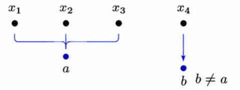

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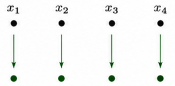

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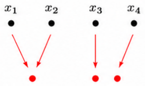

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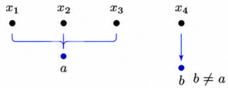

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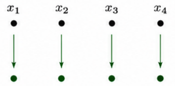

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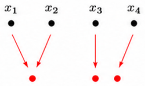

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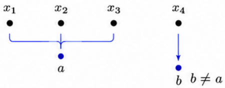

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5. Almost k-to-one case

If F is almost k-to-one (every fiber has size $k \geq 3$), then there are $\dfrac{2^n - 1}{k}$ fibers (since $k \mid (2^n - 1)$).

$$|UB_{n,F}| = \frac{2^n - 1}{k} \cdot \binom{k}{3} = \frac{(2^n - 1)(k-1)(k-2)}{6}$$

Example

Gold function: $F : \mathbb{F}_{2^6} \to \mathbb{F}_{2^6}$, $F(x) = x^3$.

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$$\begin{aligned}\mathcal{UF}_{6,F} = \{&\{38,0,24,62\},\{21,46,0,59\},\{29,0,16,13\},\\ &\{40,0,49,25\},\{51,0,54,5\},\{27,47,0,52\},\\ &\{19,0,31,12\},\{55,0,61,10\},\{35,8,0,43\},\\ &\{23,0,39,48\},\{22,41,63,0\},\{50,57,0,11\},\\ &\{14,15,0,1\},\{30,0,2,28\},\{3,0,18,17\},\\ &\{37,0,9,44\},\{56,60,0,4\},\{0,33,53,20\},\\ &\{0,32,58,26\},\{7,0,42,45\},\{36,0,6,34\}\}.\end{aligned}$$

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F is almost 3-to-1

$$|\mathcal{UF}_{6,F}| = \frac{\binom{3}{3}(2^6-1)}{3} = 21.$$

$$\begin{aligned}\mathcal{UF}_{6,F} = \{&\{38,0,24,62\},\{21,46,0,59\},\{29,0,16,13\},\\ &\{40,0,49,25\},\{51,0,54,5\},\{27,47,0,52\},\\ &\{19,0,31,12\},\{55,0,61,10\},\{35,8,0,43\},\\ &\{23,0,39,48\},\{22,41,63,0\},\{50,57,0,11\},\\ &\{14,15,0,1\},\{30,0,2,28\},\{3,0,18,17\},\\ &\{37,0,9,44\},\{56,60,0,4\},\{0,33,53,20\},\\ &\{0,32,58,26\},\{7,0,42,45\},\{36,0,6,34\}\}.\end{aligned}$$

Flat $\{x_1,x_2,x_3,x_4\}$ is represented by the integer labels of its elements in $\mathbb{F}_{2^6}$.

Question

What is the maximal dimension of an unbroken flat, given an APN function?

Some computational observations

Table: Representatives of CCZ classes of APN functions for $n = 6$ with $\deg(F) \leq 3$. α is a root of the primitive polynomial $x^6 + x^4 + x^3 + x + 1 \in \mathbb{F}_2[x]$.

F	Univariate representation of F
D_1	x^3
D_2	$x^3 + \alpha^{11}x^6 + \alpha x^9$
D_3	$\alpha x^5 + x^9 + \alpha^4 x^{17} + \alpha x^{18} + \alpha^4 x^{20} + \alpha x^{24} + \alpha^4 x^{34} + \alpha x^{40}$
D_4	$\alpha^7 x^3 + x^5 + \alpha^3 x^9 + \alpha^4 x^{10} + x^{17} + \alpha^6 x^{18}$
D_5	$x^3 + \alpha x^{24} + x^{10}$
D_6	$x^3 + \alpha^{17}(x^{17} + x^{18} + x^{20} + x^{24})$
D_7	$x^3 + \alpha^{11}x^5 + \alpha^{13}x^9 + x^{17} + \alpha^{11}x^{33} + x^{48}$
D_8	$\alpha^{25}x^5 + x^9 + \alpha^{38}x^{12} + \alpha^{25}x^{18} + \alpha^{25}x^{36}$
D_9	$\alpha^{40}x^5 + \alpha^{10}x^6 + \alpha^{62}x^{20} + \alpha^{35}x^{33} + \alpha^{15}x^{34} + \alpha^{29}x^{48}$
D_{10}	$\alpha^{34}x^6 + \alpha^{52}x^9 + \alpha^{48}x^{12} + \alpha^6 x^{20} + \alpha^9 x^{33} + \alpha^{23}x^{34} + \alpha^{25}x^{40}$
D_{11}	$x^9 + \alpha^4(x^{10} + x^{18}) + \alpha^9(x^{12} + x^{20} + x^{40})$
D_{12}	$\alpha^{52}x^3 + \alpha^{47}x^5 + \alpha x^6 + \alpha^9 x^9 + \alpha^{44}x^{12} + \alpha^{47}x^{33} + \alpha^{10}x^{34} + \alpha^{33}x^{40}$
D_{13}	$\alpha(x^6 + x^{10} + x^{24} + x^{33}) + x^9 + \alpha^4 x^{17}$
D. APN Perm.	long univariate representation
EP	$x^3 + \alpha^{17}(x^{17} + x^{18} + x^{20} + x^{24}) + \alpha^{14}(\alpha^{18}x^9 + \alpha^{36}x^{18} + \alpha^9 x^{36}$ $+x^{21} + x^{42} + \mathrm{Tr}(\alpha^{27}x + \alpha^{52}x^3 + \alpha^6 x^5 + \alpha^{19}x^7 + \alpha^{28}x^{11} + \alpha^2 x^{13}))$

Unbroken 2-flats

Table: Number of unbroken and strongly unbroken 2-flats of APN functions in 6 variables.

F	# unbroken 2-flats	# strongly unbroken 2-flats
D_1	21	0
D_2	21	0
D_3	9	13
D_4	8	13
D_5	7	21
D_6	18	9
D_7	18	15
D_8	4	9
D_9	15	17
D_{10}	8	13
D_{11}	6	13
D_{12}	7	14
D_{13}	8	11
D. APN Perm.	0	0
EP	25	21

Unbroken 3-flats

Table: Number of unbroken and strongly unbroken 3-flats of APN functions in 6 variables.

F	# unbroken 3-flats	# strongly unbroken 3-flats
D_1	9	3285
D_2	0	3285
D_3	0	7779
D_4	1	7721
D_5	23	6561
D_6	1	7332
D_7	1	7073
D_8	0	7617
D_9	10	6640
D_{10}	1	7711
D_{11}	1	7809
D_{12}	1	7586
D_{13}	0	7502
D. APN Perm.	0	184
EP	0	4336

Unbroken 4-flats

Table: Number of unbroken and strongly unbroken 4-flats of APN functions in 6 variables.

F	# unbroken 4-flats	# strongly unbroken 4-flats
D_1	0	714
D_2	0	714
D_3	0	546
D_4	0	549
D_5	0	203
D_6	0	497
D_7	0	454
D_8	0	473
D_9	0	329
D_{10}	0	531
D_{11}	0	559
D_{12}	0	494
D_{13}	0	499
D. APN Perm.	0	812
EP	0	391

Unbroken 4-flats - there's none :)

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More notation and basic notions

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Idea: Estimate the number of broken and unbroken 2-flats.

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Unbroken 2-flats: $\mathcal{UF}_{n,F} = \{L \in \mathcal{F}_{2,n} : F(L) \text{ is a flat}\}$.

Theorem

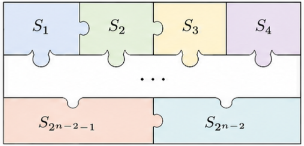
Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ and let $|\mathcal{UF}_{n,F}| = w$. Then

$$\mathcal{NL}_V(F) \geq 2^{n-2} - w.$$

Proof sketch :)

Idea: Decompose $\mathbb{F}_2^n$ into 2-dimensional flats S_i

$\mathbb{F}_2^n =$



- Partition $\mathbb{F}_2^n$ into 2^{n-2} disjoint 2-dimensional affine subspaces (flats) $S_1, S_2, \ldots, S_{2n-2}$.

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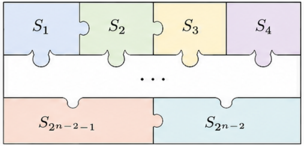
$\mathbb{F}_2^n =$ 

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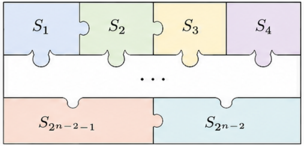

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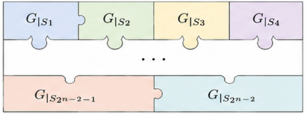

The restriction of G to each flat S_i is also affine.

Thus, choosing G is equivalent to choosing affine functions $G_{|S_i}$ on each S_i.

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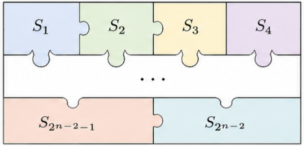

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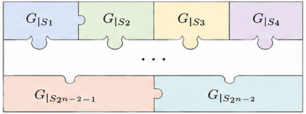

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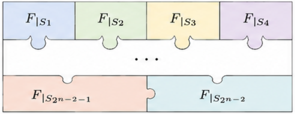

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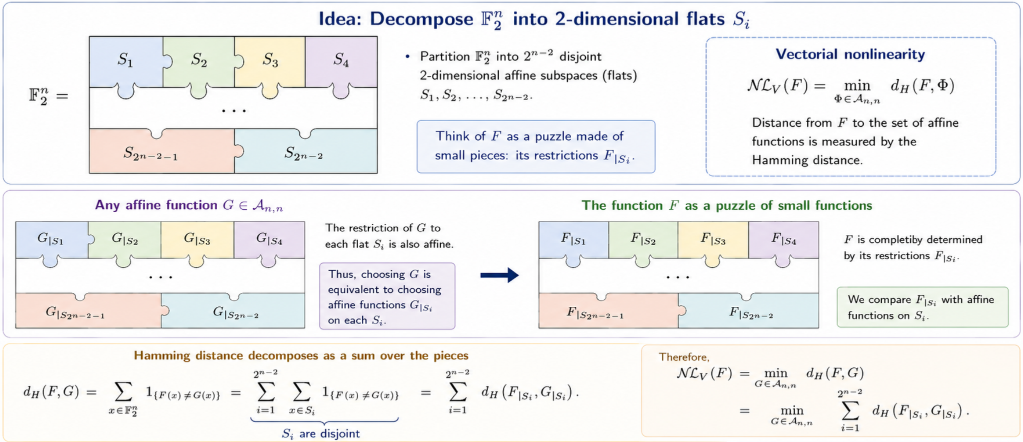

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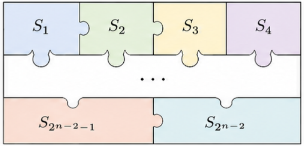
$\mathbb{F}_2^n =$ 

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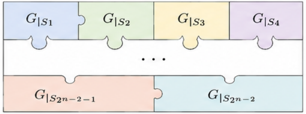

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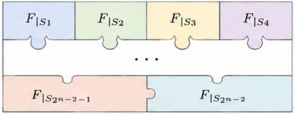

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Hamming distance decomposes as a sum over the pieces

$$d_H(F,G) = \sum_{x \in \mathbb{F}_2^n} 1_{\{F(x) \neq G(x)\}} = \underbrace{\sum_{i=1}^{2^{n-2}} \sum_{x \in S_i} 1_{\{F(x) \neq G(x)\}}}_{S_i \text{ are disjoint}} = \sum_{i=1}^{2^{n-2}} d_H\left(F_{|S_i}, G_{|S_i}\right).$$

Therefore,

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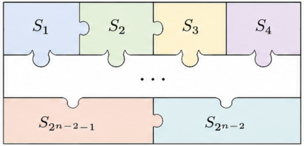

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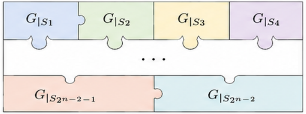

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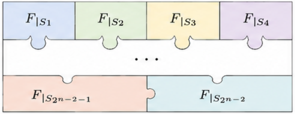

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Conclusion

If $|\mathcal{U}_{2,F}| = w$ (i.e., w unbroken flats), then at least $2^{n-2} - w$ flats are broken, so $\mathcal{NL}_V(F) \geq 2^{n-2} - w$.

Lower bound on vectorial nonlinearity

Unbroken 2**-flats:** $\mathcal{UF}_{n,F} = \{L \in \mathcal{F}_{2,n} : F(L) \text{ is a flat}\}$.

Theorem

Let $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ *and let* $|\mathcal{UF}_{n,F}| = w$. *Then*

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Corollary

Let $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ *be the inverse function defined by* $F(x) = x^{-1}$, *where* n *is odd. Then* F *breaks all 2-flats [3]. Hence*

$$\mathcal{NL}_V(F) \geq 2^{n-2}.$$

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To study this, define

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Let $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ satisfy $|F^{-1}(y)| \leq 3$ for all $y \in \mathbb{F}_{2^n}$. If L is a k-flat and $F(L)$ is an r-flat with $r \geq 2$, then

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Conclusion: An unbroken k-flat can only have as its image a $(k-1)$-flat or a k-flat.

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Observation from the airport: $\phi_F : \mathcal{F}_{2,n} \to \mathbb{F}_{2^n}$, $\phi_F(\{x_1, x_2, x_3, x_4\}) = \sum_{i=1}^{4} F(x_i)$

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$$\mathbb{F}_{2^n}^* = V_{56} \cup V_7, \quad \forall c \in V_{56} : |\phi_F^{-1}(c)| = 160 \text{ and } \forall c \in V_7 : |\phi_F^{-1}(c)| = 208$$

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Thank you! :)
Questions?

References I

L. Budaghyan, I. Ivkovic, and N. Kaleyski.
Triplicate functions.
Cryptography and Communications, 15:35–83, 2023.

C. Carlet.
Two generalizations of almost perfect nonlinearity.
Journal of Cryptology, 38(20), 2025.

N. Kolomeec and D. Bykov.
On the image of an affine subspace under the inverse function within a finite field.
Designs, Codes and Cryptography, 92(2):467–476, 2024.

L. Kölsch, B. Kriepke, and G. M. Kyureghyan.
Image sets of perfectly nonlinear maps.
Designs, Codes and Cryptography, 91:1–27, 2023.

S. Li, W. Meidl, A. Polujan, A. Pott, C. Riera, and P. Stanica.
Vanishing flats: a combinatorial viewpoint on the planarity of functions and their application.
IEEE Transactions on Information Theory, 66(11):7101–7112, 2020.