Algebraic and geometric methods in the theory of subspace codes

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Università degli Studi della Campania “Luigi Vanvitelli”

WCC 2026
The Fourteenth International Workshop on Coding and Cryptography
June 8-12, 2026, Paris (France)

Outline

Network coding

R. Ahlswede, N. Cai, S. R. Li,: *Network Information Flow*, IEEE TRANSACTIONS ON INFORMATION THEORY, **2000**.

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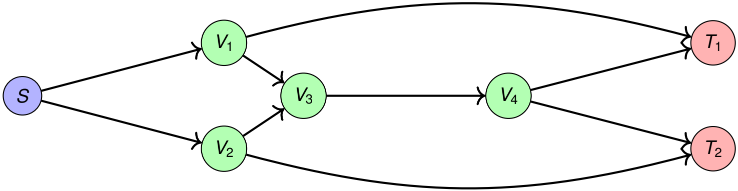

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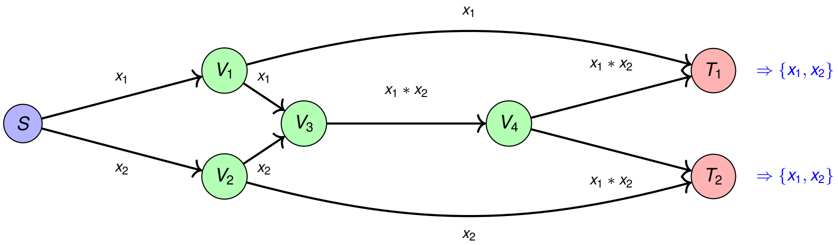
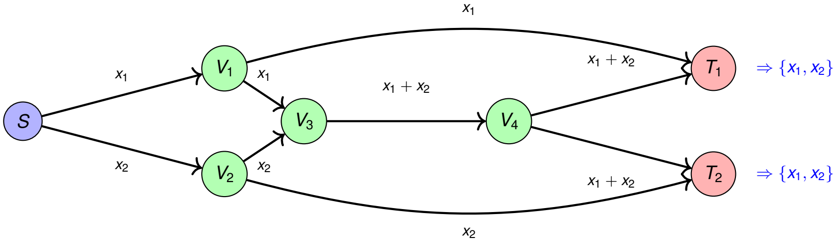


Linear Network coding

S. R. Li, R.W. Yeung, N. Cai: *Linear Network Coding*, IEEE Transactions on Information Theory, **2003**.



RLNC: Random Linear Network Coding

T. Ho, M. Médard, R. Koetter, D. R. Karger, M. Effros, J. Shi and, B. Leong: *A random linear network coding approach to multicast* , IEEE Transactions on Information Theory, **2006**.

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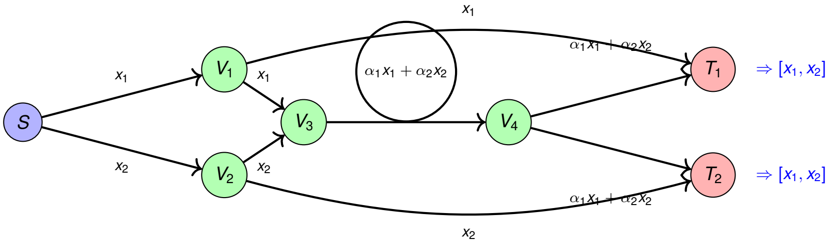

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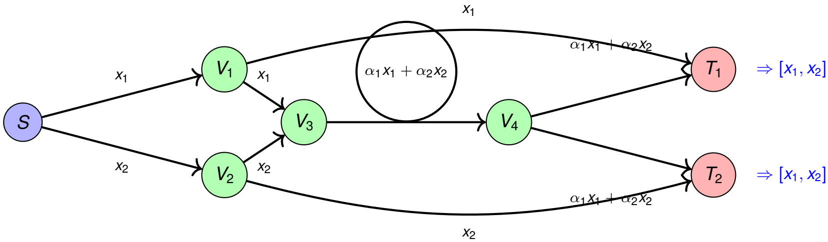

R. Koetter, and F.R. Kschischang : *Coding for errors and erasures in random network coding* , IEEE TRANSACTIONS ON INFORMATION THEORY, **2008**.

Subspace codes

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Linear isometries

$$\mathcal{C}, \mathcal{C}' \subseteq \mathcal{P}_q(n)$$

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$$A_q(n,d;k) = A_q(n,d;n-k)$$

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$$A_q(n,d;k) = A_q(n,d;n-k) \longrightarrow k \leq \frac{n}{2}$$

Some bounds for $A_q(n, d, k)$

$$A_q(n, d; k),\ \delta := \frac{d}{2},\ e := \left\lfloor \frac{\delta - 1}{2} \right\rfloor \text{ and } k \leq \frac{n}{2}$$

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S. Kurz: *Constructions and bound for subspace codes*, arXiv, **2025**.

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$A_q(n,d,k)$ **for** $n\to\infty$ **[Blackburn-Etzion** IEEE TIT, **2012]**

Let q, k, and $d=2\delta$ be fixed integers with $0\leq\delta\leq k+1$ and such that q is a prime power. Then

$$A_q(n,d,k)\sim\frac{\begin{bmatrix} n\\ k-\delta+1\end{bmatrix}_q}{\begin{bmatrix} k\\ k-\delta+1\end{bmatrix}_q}\qquad n\to\infty$$

Optimal Subspace Codes

$$A_q(n,d;k) \stackrel{?}{=} \frac{\begin{bmatrix} n \\ k-\frac{d}{2}+1 \end{bmatrix}_q}{\begin{bmatrix} k \\ k-\frac{d}{2}+1 \end{bmatrix}_q}$$

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Spread Codes

$$d = 2k$$

Optimal Subspace Codes

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Spread Codes

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Spreads of projective spaces $\longrightarrow$ translation structures

B. Segre: *Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane*
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J. André: *Über nicht-Desarguessche Ebenen mit transitive Translationgruppe.* MATH. Z., **1954**.

R.H. Bruck, R.C. Bose: *The construction of translation planes from projective spaces* J. ALGEBRA, **1964**.

H. Lübeburg: *Translation Planes,* BERLIN: SPRINGER VERLAG, **1980**.

M. Biliotti, V. Jha, N. L. Johnson: *Foundations of Translation Planes,* MARCEL DEKKER, **2001**.

G. Lunardon: *50 Years of translation structures,* JOURNAL OF GEOMETRY, **2022**.

q-Steiner systems

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M. Schwartz, T. Etzion: *Codes and Anticodes in the Grassman Graph*, JCTA, **2002**.

q-Steiner systems: known examples

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A. Beutelspacher: *Parallelismen in unendlichen projektiven Raumen endlicher Dimension*, GEOMETRIAE DEDICATA, **1978**.

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There exist $\mathcal{S}_2(2, 3, 13)$ 2-systems $\longleftrightarrow$ $A_2(13, 4, 3) = \dfrac{\begin{bmatrix} 13 \\ 2 \end{bmatrix}_2}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}_2} = 1597245$

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$$\mathcal{C} = \cup_V Orb_G(V) = \cup_V \{\varphi(V) \ : \ \varphi \in G\}$$

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Desarguesian spread code

F. Manganiello, E. Gorla, J. Rosenthal *Spread Codes and Spread Decoding in Network Coding,* IEEE TIT, **2013**.

E. Gorla, F. Manganiello: *An Algebraic Approach for Decoding Spread Codes* Advances in Mathematics of Communications , **2011**.

E. Franch, C. Li, A. Piccirillo: *Decoding Desarguesian spread codes beyond half minimum distance* Man,**2026**.

Recent results on cyclic subspace codes

GGA

Galois geometries and their applications

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C. Castello, O.P., F. Zullo: *On one-orbit cyclic subspace codes of $\mathcal{G}_q(n,3)$* , IEEE, ISIT PROCEEDINGS, **2024**.

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- **C. Castello, O.P., P. Santonastaso. F. Zullo:** *Construction and equivalence of Sidon sapces* , J. ALGEBRAIC COMB., **2023**.
- **F. Zullo:** *Multi-orbit cyclic subspace codes and linear sets*, JFINITE FIELDS AND THEIR APPLICATIONS, **2023**.
- **C. Castello, H. Gluesing-Luerssen, O.P., F. Zullo:** *Quasi-optimal cyclic orbit codes*, DESIGNS, CODES AND CRYPTOGRAPHY, **2026**.
- **C. Castello, P. Santonastaso:** *Asymptotically optimal cyclic subspace codes*, *arXiv.2507.09290*, **2025**.

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Conjecture [A.-L. Trautmann, F. Manganiello, M. Braun and J. Rosenthal, IEEE TIT, **2013.]**

We conjecture that for any prime power q and positive integers k and n, we can construct optimal cyclic subspace codes $\mathcal{C} \subset \mathcal{G}_q(n,k)$, i.e. $d(\mathcal{C}) = 2k-2$ and $|\mathcal{C}| = \frac{q^n-1}{q-1}$.

Construction of optimal cyclic orbit codes

- **A.-S. Elsenhans, A. Kohnert** *Constructing network codes using Möbius transformations*, PREPRINT, **2010**, $\rightarrow$ $q = 2$, $n \geq 4k - 6$
- **E. Ben-Sasson, T. Etzion, A. Gabizon and N. Raviv:** *Subspace polynomials and cyclic subspace codes*, IEEE TRANSACTION ON INFORMATION THEORY, **2016**. $\rightarrow$ k, q and infinitely many values of n
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Property [Roth, Raviv,Tamo: IEEE TIT, 2018]

$$S \in \mathcal{G}_q(n,k),\ S \leq_{\mathbb{F}_q} \mathbb{F}_{q^n}$$

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Sidon Spaces in Finite Fields

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If $q > 2$ and $n \geq \frac{k(k+1)}{2}$ (or $q = 2$ and $n > \frac{k(k+1)}{2}$), then a k-dimensional max span Sidon space in $\mathbb{F}_{q^n}$ can be found in polynomial expected running time by a probabilistic algorithm.

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R.M. Roth, N. Raviv and I. Tamo: *Construction of Sidon spaces with applications to coding*, IEEE Transaction on Information Theory, **2018**.

$$S = \{u + u^{q^s}\gamma \ : \ u \in \mathbb{F}_{q^k}\} \leq_{\mathbb{F}_q} \mathbb{F}_{q^n}$$

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Corollary

$f \in \mathcal{L}_{k,q}[x]$ Sidon polynomial $\iff$
$\mathcal{C}_{f,\gamma} = Orb(S_{f,\gamma})$ optimal cyclic orbit code in $\mathcal{G}_q(kr, k)$ $\forall r \geq 3$

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Theorem

If n is even, then there exist quasi-optimal full-length orbit codes in $\mathcal{G}_q(n,k)$ for every $3 \leq k \leq \frac{n}{2}$ and for every $q \geq 2$.

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Thank you

Thank you for your attention!