# Generalized Sum Rank Metric for Convolutional Codes

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# Block codes

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$t = 1$

**Sum Rank metric block codes**

$$\mathcal{C} \leq \mathbb{F}_q^{n_1 \times m_1} \times \cdots \times \mathbb{F}_q^{n_t \times m_t}$$

$$wt_{SR}((C_1, \ldots, C_t)) := rank(C_1) + \cdots + rank(C_t)$$

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submodule $\mathcal{C} \subseteq \mathbb{F}_q[z]^n$

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**Generalized sum rank metric convolutional codes**

# Generalized sum rank metric

Let $t, m_j, n_j \in \mathbb{N}_0, j = 1, \ldots, t$. For

$$A(z) = \sum_{i \in \mathbb{N}_0} (A_{1,i}, \ldots, A_{t,i}) z^i \in R := \mathbb{F}_q[z]^{n_1 \times m_1} \times \cdots \times \mathbb{F}_q[z]^{n_t \times m_t}$$

the **generalized sum rank weight** is given by

$$wt_{SR}(A(z)) := \sum_{j=1}^{t} \sum_{i \in \mathbb{N}_0} rank(A_{j,i}).$$

For $B(z) = \sum_{i \in \mathbb{N}_0} (B_{1,i}, \ldots, B_{t,i}) z^i \in R$ the **generalized sum rank distance** between $A(z)$ and $B(z)$ is defined as

$$d_{SR}(A(z), B(z)) := wt_{SR}(A(z) - B(z)) = \sum_{j=1}^{t} \sum_{i \in \mathbb{N}_0} rank(A_{j,i} - B_{j,i}).$$

# Generalized sum rank metric convolutional code

A **generalized sum rank metric convolutional code** $\mathcal{C}$ is the image of a homomorphism

$$\begin{aligned}\varphi : \mathbb{F}_q[z]^k &\xrightarrow{\psi} \mathbb{F}_q[z]^{n_1 m_1 + \cdots + n_t m_t} \xrightarrow{\gamma} \mathbb{F}_q[z]^{n_1 \times m_1} \times \cdots \times \mathbb{F}_q[z]^{n_t \times m_t} \\ m(z) &\mapsto c(z) = m(z)G(z) \mapsto C(z)\end{aligned}$$

where $G(z) \in \mathbb{F}_q[z]^{k \times (n_1 m_1 + \cdots + n_t m_t)}$ is a polynomial matrix of full rank, called a **generator matrix** of $\mathcal{C}$.

The **generalized sum rank distance** of $\mathcal{C}$ is

$$d_{SR}(\mathcal{C}) = \min_{\substack{A(z), B(z) \in \mathcal{C}, \\ A(z) \neq B(z)}} d_{SR}(A(z), B(z)) = \min_{C(z) \in \mathcal{C} \setminus \{0\}} wt_{SR}(C(z)).$$

# Generalized sum rank metric convolutional code

The **degree** $\delta$ of a convolutional code $\mathcal{C}$ is defined as the maximal degree of the $k \times k$ minors of a generator matrix $G(z)$.

The $i$-th row degree $\nu_i$ of $G(z)$ is the largest degree of any entry in row $i$ of $G(z)$. If the sum of all row degrees is equal to $\delta$ we call $G(z)$ a **minimal** generator matrix.

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A generalized sum rank-metric convolutional code $\mathcal{C} \subseteq \mathbb{F}_q[z]^{n_1 \times m_1} \times \cdots \times \mathbb{F}_q[z]^{n_t \times m_t}$ with degree $\delta$ and dimension $k$ is denoted as a $((n_1 \times m_1, \ldots, n_t \times m_t), k, \delta)$ convolutional code.

# Example

Consider an $((1 \times 2, 2 \times 2), 2, 2)$ convolutional code over $\mathbb{F}_2$ with

$$G(z) = \left( \begin{array}{cc:cccc} 1 & z & 1 & z & z & 1+z \\ z & 1+z & z & 1 & 1 & 1+z \end{array} \right).$$

For a message $m(z) = \begin{pmatrix} 1 & 0 \end{pmatrix}$ we get

$$c(z) = m(z)G(z) = \left( \begin{array}{cc:cccc} 1 & z & 1 & z & z & 1+z \end{array} \right)$$

and

$$C(z) = \left( \begin{pmatrix} 1 & z \end{pmatrix}, \begin{pmatrix} 1 & z \\ z & 1+z \end{pmatrix} \right) = \left( \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) + \left( \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right) z$$

which has weight 6.

# Singleton bound for generalized sum rank metric convolutional codes

**Theorem (Abreu, -, Lieb, Pinto, 2026)**

*Let $\mathcal{C}$ be an $((n_1 \times m_1, \ldots, n_t \times m_t), k, \delta)$ generalized sum rank metric convolutional code. Let $\alpha \in \{1, \ldots, t\}$ be the maximum value such that*

$$\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) k - \delta - 1 \geq n_1 m_1 + \cdots + n_{\alpha-1} m_{\alpha-1}$$

*and $n = n_1 + \cdots + n_t$. Then,*

$$d_{SR}(\mathcal{C}) \leq n \left\lfloor \frac{\delta}{k} \right\rfloor + \sum_{j=\alpha}^{t} n_j - \left\lfloor \frac{(\lfloor \frac{\delta}{k} \rfloor + 1)k - \delta - 1 - \sum_{j=1}^{\alpha-1} n_j m_j}{m_\alpha} \right\rfloor .$$

# Singleton bounds for convolutional codes

**Hamming metric** [1]

$$d_H(\mathcal{C}) \leq (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1$$

**Rank metric** [2]

$$d_R(\mathcal{C}) \leq n\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) - \left\lceil \frac{(\lfloor \frac{\delta}{k} \rfloor + 1)k - \delta}{m} \right\rceil + 1$$

$$n_i = m_i = 1$$
$$\alpha = \left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) k - \delta$$

$$t = 1$$
$$\alpha = 1$$

**Generalized sum rank metric**

$$d_{SR}(\mathcal{C}) \leq n\left\lfloor \frac{\delta}{k} \right\rfloor + \sum_{j=\alpha}^{t} n_j - \left\lfloor \frac{(\lfloor \frac{\delta}{k} \rfloor + 1)k - \delta - 1 - \sum_{j=1}^{\alpha-1} n_j m_j}{m_\alpha} \right\rfloor$$

[1] Rosenthal, Smarandache, 1999

[2] Napp, Pinto, Rosenthal, Vettori, 2017

# Construction of MSRD convolutional codes

If a generalized sum rank metric convolutional code attains the Singleton-like bound we call it **Maximum Sum Rank Distance (MSRD)** convolutional code.

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**Theorem (Abreu, -, Lieb, Pinto, 2026)**

*For $j = 1, \ldots, t$, let $G^j(z) \in \mathbb{F}_q[z]^{k \times n_j m_j}$ be minimal generator matrices of $((n_j \times m_j), k, \delta)$ rank metric convolutional codes $\mathcal{C}_j$ that attain the Singleton-like bound with $d_R(\mathcal{C}_j) = n_j(\lfloor \frac{\delta}{k} \rfloor + 1)$ and the same order of row degrees.*
*Let $\mathcal{C}$ be the generalized sum rank metric convolutional code generated by*

$$G(z) = \begin{pmatrix} G^1(z) & G^2(z) & \cdots & G^t(z) \end{pmatrix}.$$

*If $\alpha = 1$, then $\mathcal{C}$ is an $((n_1 \times m_1, \ldots, n_t \times m_t), k, \delta)$ MSRD code.*

# Column sum rank distance

A convolutional code $\mathcal{C}$ is called **non-catastrophic** if $rank(G(\lambda)) = k$ for all $\lambda \in \overline{\mathbb{F}}_q$, where $\overline{\mathbb{F}}_q$ denotes the algebraic closure of $\mathbb{F}_q$.
For $j \in \mathbb{N}_0$, the **j-th column sum rank distance** of a non-catastrophic $((n_1 \times m_1, \dots, n_t \times m_t), k, \delta)$ generalized sum rank metric convolutional code is

$$d_j^{csr}(\mathcal{C}) := \min\{wt_{SR}(C(z)_{|[0,j]}) \mid C(z) \in \mathcal{C} \text{ and } (C_{1,0}, \dots, C_{t,0}) \neq 0\}$$

where

$$C(z)_{|[0,j]} = \sum_{i=0}^{j} (C_{1,i}, \dots, C_{t,i}) z^i$$

for

$$C(z) = \sum_{i \in \mathbb{N}_0} (C_{1,i}, \dots, C_{t,i}) z^i.$$

# Bounds for the column sum rank distance

**Theorem (Abreu, -, Lieb, Pinto, 2026)**

*Let $\mathcal{C}$ be an non-catastrophic $((n_1 \times m_1, \ldots, n_t \times m_t), k, \delta)$ generalized sum rank metric convolutional code. Let $\beta_1, \beta_2 \in \{1, \ldots, t\}$ be the maximal values such that*

$$k - 1 - \sum_{i=1}^{\beta_1 - 1} n_i m_i \geq 0 \quad \text{and} \quad k - \sum_{i=1}^{\beta_2 - 1} n_i m_i \geq 0.$$

*Then,*

$$d_j^{csr}(\mathcal{C}) \leq \sum_{i=\beta_1}^{t} n_i - \left\lfloor \frac{k - 1 - \sum_{i=1}^{\beta_1 - 1} n_i m_i}{m_{\beta_1}} \right\rfloor + j \left( \sum_{i=\beta_2}^{t} n_i - \left\lfloor \frac{k - \sum_{i=1}^{\beta_2 - 1} n_i m_i}{m_{\beta_2}} \right\rfloor \right).$$

# Bounds for the column distances

j-th column distance [3]

$$d_j^c(\mathcal{C}) \leq (n-k)(j+1)+1$$

j-th column rank distance [4]

$$d_j^{cr}(\mathcal{C}) \leq j\left(n - \left\lfloor \frac{k}{m} \right\rfloor\right) + n - \left\lfloor \frac{k-1}{m} \right\rfloor$$

$n_i = m_i = 1$

$\beta_1 = k, \beta_2 = k+1$

$t = 1$

$\beta_1 = \beta_2 = 1$

j-th column sum rank distance

$$d_j^{csr}(\mathcal{C}) \leq \sum_{i=\beta_1}^{t} n_i - \left\lfloor \frac{k-1-\sum_{i=1}^{\beta_1-1} n_i m_i}{m_{\beta_1}} \right\rfloor + j\left(\sum_{i=\beta_2}^{t} n_i - \left\lfloor \frac{k-\sum_{i=1}^{\beta_2-1} n_i m_i}{m_{\beta_2}} \right\rfloor\right)$$

[3] Gluesing-Luerssen, Rosenthal, Smarandache, 2006

[4] Napp, Pinto, Rosenthal, Santana, 2017

# Construction of rank metric convolutional codes

A convolutional code with a minimal generator matrix $G(z) = \sum_{i=0}^{\mu} G_i z^i$ where $G_\mu \neq 0$ has **memory** $\mu$.

If the bound for the j-th column sum rank distance is achieved for $j = \mu$, then we call the code **Memory Maximum Sum Rank (m-MSR)**.
If this bound is achieved for

$$L = \left\lfloor \frac{n \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta - k\left( \left\lfloor \frac{\delta}{k} \right\rfloor + 1 \right)}{m} \right\rfloor + \left\lfloor \frac{k-1}{m} \right\rfloor + 1}{n - \left\lfloor \frac{k}{m} \right\rfloor} \right\rfloor,$$

we call the code **Maximal Rank Profile (MRP)**.

# Construction of rank metric convolutional codes

Assume $\left\lfloor \frac{k}{m} \right\rfloor = \left\lfloor \frac{k-1}{m} \right\rfloor$. Let $\mathcal{D} \subseteq \mathbb{F}_q^{n\times m}$ be an rank metric block code with rank distance $d_R(\mathcal{D}) = n - \left\lfloor \frac{k}{m} \right\rfloor$ and dimension $dim(\mathcal{D}) = m(\left\lfloor \frac{k}{m} \right\rfloor + 1)$. We can choose a basis $\{A_1, \ldots, A_{m(\left\lfloor \frac{k}{m} \right\rfloor + 1)}\}$ of $\mathcal{D}$ and set

$$G_i = \begin{pmatrix} \psi^{-1}(A_{ik+1}) \\ \psi^{-1}(A_{ik+2}) \\ \vdots \\ \psi^{-1}(A_{(i+1)k}) \end{pmatrix} \quad \text{for} \quad i = 0, \ldots, j \leq \left\lfloor \frac{m(\left\lfloor \frac{k}{m} \right\rfloor + 1)}{k} \right\rfloor - 1.$$

Let $\mathcal{C} \subseteq \mathbb{F}_q^{n\times m}$ be the $k$-dimensional rank metric convolutional code with $\delta = kj$ and generator matrix $G(z) = \sum_{i \in \mathbb{N}_0} G_i z^i$.

# Construction of rank metric convolutional codes

## Example:

Let $n = m = 2$ and $k = 1$. Consider the rank metric block code with basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

over $\mathbb{F}_2$ which has minimum distance $d_R(\mathcal{D}) = 2 = n - \lfloor \frac{k}{m} \rfloor$ and dimension $dim(\mathcal{D}) = 2 = m(\lfloor \frac{k}{m} \rfloor + 1)$.
Then

$$G_0 = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$
$$G_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}$$

and therefore

$$G(z) = G_0 + G_1 z = \begin{pmatrix} 1 + z & z & z & 1 \end{pmatrix}.$$

# Construction of rank metric convolutional codes

**Theorem (Abreu, -, Lieb, Pinto, 2026)**

*The $k$-dimensional rank metric convolutional code $\mathcal{C} \subseteq \mathbb{F}_q^{n\times m}$ with $\delta = kj$ and generator matrix $G(z) = \sum_{i\in\mathbb{N}_0} G_i z^i$ is an m-MSR and an MRP code.*

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## Thank you for your attention