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Optimal Multidimensional Convolutional Codes

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Outline

mD Convolutional codes

- Coding theory $\Rightarrow$ **Convolutional Codes**
- Convolutional codes are especially useful for **sequential encoding** and **decoding with low delay** and hence very important for streaming applications.

mD Convolutional codes

- Coding theory ⇒ **Convolutional Codes**
- Convolutional codes are especially useful for **sequential encoding** and **decoding with low delay** and hence very important for streaming applications.

Muiltidimensional (mD) convolutional codes generalize convolutional codes to polynomial rings in several variables, making them **ideal for transmitting multidimensional data**.

- Fornasini and Valcher introduced 2D convolutional codes [1, 2]. Later works established free distance upper bounds [3] and proposed various optimal constructions [3, 4, 5, 6].
- Significant differences exist between 1D and 2D codes, and between 2D and 3D codes [7].

mD Convolutional Codes

Let $\mathbb{F}$ be a finite field and $R = \mathbb{F}[z_1, \ldots, z_m]$ the polynomial ring in m variables with coefficients in $\mathbb{F}$.

Definition ([7])

An **mD** (finite support) **convolutional code** $\mathcal{C}$ of rate k/n is a free R-submodule of R^n of rank k.

A full row rank matrix $G(z_1, \ldots, z_m) \in R^{k \times n}$ whose rows constitute a basis for $\mathcal{C}$ is called an **encoder** of $\mathcal{C}$ and therefore

$$\begin{aligned}\mathcal{C} &= \mathrm{Im}_R\, G(z_1, \ldots, z_m) \\ &= \{v(z_1, \ldots, z_m) \in R^n : v(z_1, \ldots, z_m) = G(z_1, \ldots, z_m)u(z_1, \ldots, z_m), u(z_1, \ldots, z_m) \in R^k\}.\end{aligned}$$

Equivalent encoders

Two full row rank matrices $G_1(z_1,\ldots,z_m), G_2(z_1,\ldots,z_m) \in R^{k\times n}$ are said to be **equivalent encoders** if

$$\mathrm{Im}_R\, G_1(z_1,\ldots,z_m) = \mathrm{Im}_R\, G_1(z_1,\ldots,z_m),$$

which happens if and only if $G_1(z_1,\ldots,z_m) = G_2(z_1,\ldots,z_m)U(z_1,\ldots,z_m)$ for some unimodular matrix $U(z_1,\ldots,z_m) \in R^{k\times k}$.

Definition ([7])

A square matrix $U(z_1,\ldots,z_m) \in R^{k\times k}$ is **unimodular** if and only if there is a matrix $V(z_1,\ldots,z_m) \in R^{k\times k}$ such that

$$U(z_1,\ldots,z_m)\cdot V(z_1,\ldots,z_m) = V(z_1,\ldots,z_m)\cdot U(z_1,\ldots,z_m) = I_k.$$

- A matrix $U(z_1,\ldots,z_m) \in R^{k\times k}$ is unimodular if and only if $det(U(z_1,\ldots,z_m))$ is a unit in R, i.e. a nonzero element of $\mathbb{F}$.

Notation

Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$. By z^α we mean $z_1^{\alpha_1} \cdot \dots \cdot z_m^{\alpha_m}$.

A polynomial $f \in R$ can be written as

$$f(z_1, \dots, z_m) = \sum_{\alpha \in \mathbb{N}^m} f_\alpha z^\alpha \text{ where } f_\alpha \in \mathbb{F}.$$

In a similar way, a vector $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \in R^n$ can be written as

$$w = \sum_{\alpha \in \mathbb{N}^m} w_\alpha z^\alpha \text{ where } w_\alpha \in \mathbb{F}^n.$$

Weight

Definition

The weight of $a \in \mathbb{F}^n$ is denoted by $wt(a)$ and is given by the number of nonzero entries of a. The **weight of** $f \in R$ is denoted by $wt(f)$ and is the **number of nonzero terms of** f. If $w = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix} \in R^n$ then the weight of w is given by

$$wt(w) = \sum_{j=1}^{n} wt(w_j).$$

Equivalently, if $w = \sum_{\alpha \in \mathbb{N}^m} w_\alpha z^\alpha$ where $w_\alpha \in \mathbb{F}^n$ then

$$wt(w) = \sum_{\alpha \in \mathbb{N}^m} wt(w_\alpha).$$

Example

Let $\mathbb{F} = \mathbb{F}_2$ and $R = \mathbb{F}[z_1, z_2]$. If $w = \begin{bmatrix} 1 + z_2 & z_1 z_2 + z_2 & z_1^2 \end{bmatrix}$, then $w = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} z_2 + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} z_1^2 + \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} z_1 z_2$. Thus $wt(w) = 5$.

Distance

Definition ([7])

Given two elements $w, \tilde{w} \in R^n$, the (Hamming) distance between w and $\tilde{w}$ is given by $dist(w, \tilde{w}) = wt(w - \tilde{w})$. The **free distance** of $\mathcal{C}$ is

$$dist(\mathcal{C}) = min\{dist(w, \tilde{w}) : w, \tilde{w} \in \mathcal{C}, w \neq \tilde{w}\}.$$

For any convolutional code $\mathcal{C}$, since $dist(w_1, w_2) = wt(w_1 - w_2)$ and $\mathcal{C}$ is linear, then

$$dist(\mathcal{C}) = \min\{wt(w) : w \in \mathcal{C}, w \neq 0\}.$$

Degree

The degree of polynomial

$$f = \sum_{\alpha \in \mathbb{N}^m} f_\alpha z^\alpha \in R$$

is, as usual, defined by the formula

$$max\{\alpha_1 + \cdots + \alpha_m : f_\alpha \neq 0\}.$$

Let ν_i be the **column degree** of the i-th column of a polynomial matrix $G(z_1, \ldots, z_m)$, i.e, the **maximum degree of the entries of the i-th column of** $G(z_1, \ldots, z_m)$. The external degree of $G(z_1, \ldots, z_m)$ is the sum of its column degrees, i.e., $\sum_{i=1}^{k} \nu_i$.

Definition

The **degree** of $\mathcal{C}$ is defined as the minimum of the external degrees among all the encoders of $\mathcal{C}$.

mD Generalized Singleton Bound

- Upper bound on the distance of mD convolutional codes of rate k/n and degree δ.

Lemma ([8])

Let $\#S$ denote the cardinality of S. Then

$$\#\{\alpha \in \mathbb{N}_0^m : 1 \leq \alpha_1 + \cdots + \alpha_m \leq \nu\} = \frac{(\nu + m)!}{\nu! m!} - 1$$

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Theorem

Let $\mathcal{C}$ be a mD convolutional code of rate k/n and degree δ. Then

$$dist(\mathcal{C}) \leq n\frac{(\lfloor \frac{\delta}{k} \rfloor + m)!}{\lfloor \frac{\delta}{k} \rfloor! m!} - k\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1.$$

Proof of the Theorem

Let $G(z_1, \ldots, z_m) \in R^{n \times k}$ be an encoder of $\mathcal{C}$ with column degrees $\nu_1, \nu_2, \ldots, \nu_k$ and external degree δ. Then $\nu_1 + \nu_2 + \cdots + \nu_k = \delta$, and let us assume that

$$\nu_1 \geq \nu_2 \geq \cdots > \nu_t = \nu_{t+1} = \nu_{t+2} = \cdots = \nu_k,$$

i.e. ν_k is the minimum value of the column degrees of $G(z_1, \ldots, z_m)$ and $G(z_1, \ldots, z_m)$ has at least $k - t + 1$ columns degrees equal to ν_k, for $1 \leq t \leq k$.

Let us write

$$G(z_1, \ldots, z_m) = \begin{bmatrix} G^{(1)}(z_1, \ldots, z_m) & G^{(2)}(z_1, \ldots, z_m) \end{bmatrix}$$

where $G^{(1)}(z_1, \ldots, z_m) \in R^{n \times (t-1)}$ and $G^{(2)}(z_1, \ldots, z_m) \in R^{n \times (k-t+1)}$.

Since the column degrees of $G^{(2)}(z_1, \ldots, z_m)$ are all equal to ν_k we can write

$$G^{(2)}(z_1, \ldots, z_m) = \sum_{0 \leq \alpha_1 + \cdots + \alpha_m \leq \nu_k} G^{(2)}_{\alpha_1 \cdots \alpha_m} z^{(\alpha_1, \cdots, \alpha_m)}.$$

Proof of the Theorem

Let $u = \begin{bmatrix} 0 \\ \tilde{u} \end{bmatrix}$, with $\tilde{u} \in \mathbb{F}^{k-t+1}$, be a nonzero vector such that $G^{(2)}_{0\cdots 0}\tilde{u}$ has $k-t$ entries equal to zero.

Then, by the Lemma we have that:

$$\begin{aligned} wt(G(z_1,\ldots,z_m)u) &= wt(G^{(2)}(z_1,\ldots,z_m)\tilde{u}) \\ &= wt\big(G^{(2)}_{0\cdots 0}\tilde{u}\big) + \sum_{1\leq \alpha_1+\cdots+\alpha_m \leq \nu_k} wt\big(G^{(2)}_{\alpha_1\cdots\alpha_m}\tilde{u}\big) \\ &\leq n-(k-t)+n\Big(\frac{(\nu_k+m)!}{\nu_k!m!}-1\Big) \\ &= n\Big(\frac{(\nu_k+m)!}{\nu_k!m!}\Big)-(k-t) \end{aligned}$$

and therefore

$$dist(\mathcal{C}) \leq n\frac{(\nu_k+m)!}{\nu_k!m!}-(k-t)+1.$$

The maximum value of this upper bound is achieved by maximizing ν_k and then t, i.e. when $\nu_k = \left\lfloor \frac{\delta}{k} \right\rfloor$ and $t = \delta - k\left\lfloor \frac{\delta}{k} \right\rfloor + 1$ and therefore

$$\boxed{dist(\mathcal{C}) \leq n\frac{(\lfloor \frac{\delta}{k} \rfloor+m)!}{\lfloor \frac{\delta}{k} \rfloor!m!} - k\Big(\left\lfloor \frac{\delta}{k} \right\rfloor+1\Big)+\delta+1.}$$

MDS mD convolutional code

- The upper bound given is the extension to mD convolutional codes of the generalized Singleton bound for 2D convolutional codes [3] and we call it **mD generalized Singleton bound**.

- An mD convolutional code of rate k/n and degree δ with distance equal to the mD generalized Singleton bound is called a **Maximum Distance Separable (MDS) mD convolutional code**.

Construction of MDS 3D convolutional codes of rate $1/n$ and degree $\delta \leq 2$

The next definition defines superregular matrix which will be useful on the construction of these codes.

Definition

Given a square matrix $A = [a_{ij}] \in \mathbb{F}_q^{r\times r}$, define

$$\bar{a}_{ij} = \begin{cases} 0 & \text{for } a_{ij} = 0 \\ x_{ij} & \text{for } a_{ij} \neq 0 \end{cases}$$

where $X = \{x_{ij} \ : \ i, j \in \{1, \dots, r\}\}$ is a set of indeterminates over $\mathbb{F}_q$, and let $\overline{A} = [\bar{a}_{ij}] \in \mathbb{F}_q[X]$. Then A has a non trivially zero determinant if the determinant of $\overline{A}$ is not the zero polynomial. A matrix is **superregular** if each of its non trivially zero minors is nonzero.

Construction of MDS 3D convolutional codes of rate $1/n$ and degree $\delta \leq 2$

The next theorem gives sufficient conditions in order for a matrix to be superregular.

Theorem ([9])

Let α be a primitive element of a finite field $\mathbb{F} = \mathbb{F}_{p^N}$ and $B = [\nu_{i\ell}]$ be a matrix over $\mathbb{F}$ with the following properties:

1. *if $\nu_{i\ell} \neq 0$ then $\nu_{i\ell} = \alpha^{\beta_{i\ell}}$ for a positive integer $\beta_{i\ell}$;*
2. *if $\nu_{i\ell} = 0$ then $\nu_{i'\ell} = 0$, for any $i' > i$ or $\nu_{i\ell'} = 0$, for any $\ell' < \ell$;*
3. *if $\ell < \ell'$, $\nu_{i\ell} \neq 0$ and $\nu_{i\ell'} \neq 0$ then $2\beta_{i\ell} \leq \beta_{i\ell'}$;*
4. *if $i < i'$, $\nu_{i\ell} \neq 0$ and $\nu_{i'\ell} \neq 0$ then $2\beta_{i\ell} \leq \beta_{i'\ell}$.*

Suppose N is greater than any exponent of α appearing as a nontrivial term of any minor of B. Then B is superregular.

The next result is trivial and will be very useful.

Lemma

Let B be a superregular matrix and $\tilde{B}$ a matrix obtained from B by permutation of columns. Then $\tilde{B}$ is also a superregular matrix.

Construction of MDS 3D convolutional codes of rate $1/n$ and degree $\delta \leq 2$

Let us consider $\delta = 2$ and let us construct an **MDS 3D convolutional code of rate** $1/n$ **and degree** 2. An encoder $G(z_1, z_2, z_3)$ of $\mathcal{C}$ can be written as

$$G(z_1, z_2, z_3) = \sum_{0 \leq i+j+l \leq 2} G_{ijl} z_1^i z_2^j z_3^l, \text{ with } G_{ijl} \in \mathbb{F}^{n \times 1}.$$

We can write

$$\boxed{G(z_1, z_2, z_3) = G^{(0)}(z_1, z_2) + G^{(1)}(z_1, z_2) z_3 + G^{(2)}(z_1, z_2) z_3^2}$$

where

$$G^{(0)}(z_1, z_2) = \sum_{0 \leq i+j \leq 2} G_{ij0} z_1^i z_2^j,$$

$$G^{(1)}(z_1, z_2) = \sum_{0 \leq i+j \leq 1} G_{ij1} z_1^i z_2^j \text{ and}$$

$$G^{(2)}(z_1, z_2) = G_{002}.$$

Construction of MDS 3D convolutional codes of rate $1/n$ and degree $\delta \leq 2$

Theorem ([4])

Let α be a primitive element of a finite field with p^N elements $\mathbb{F} = \mathbb{F}_{p^N}$ for some $N \in \mathbb{N}$. Let $n, k, \delta, \tilde{\nu}$ such that $k \nmid \delta$, $\tilde{\nu} = \left\lfloor \frac{\delta}{k} \right\rfloor$ and $n > k + \delta$. Consider

$$G(z_1, z_2) = \sum_{0 \leq a+b \leq \tilde{\nu}+1} G_{ab} z_1^a z_2^b \in \mathbb{F}[z_1, z_2]^{n \times k}$$

with

$$G_{ab} = \left[g_{i,j}^{(a,b)} \right] \in \mathbb{F}^{n \times k}$$

defined by

$$g_{i,j}^{(a,b)} = \begin{cases} \alpha^{2^{(a(\tilde{\nu}+2)+b)n+i+j-2}} & \text{if } 0 \leq a+b \leq \tilde{\nu} \\ \alpha^{2^{(a(\tilde{\nu}+2)+b)n+i+j-2}} & \text{if } a+b = \tilde{\nu}+1 \text{ and } j \leq \delta - k\tilde{\nu} \\ 0 & \text{if } a+b = \tilde{\nu}+1 \text{ and } j > \delta - k\tilde{\nu} \\ 0 & \text{if } a+b > \tilde{\nu}+1. \end{cases}$$

Then, for $N \in \mathbb{N}$ sufficiently large, $\mathcal{C} = \mathrm{Im}_{\mathbb{F}[z_1,z_2]} G(z_1, z_2)$ is an ***MDS 2D convolutional code of rate k/n and degree δ****.*

Construction of MDS 3D convolutional codes of rate $1/n$ and degree $\delta \leq 2$

Theorem ([6])

Let n and δ be non-negative integers and set $\ell = \frac{(\delta+1)(\delta+2)}{2}$. Let $\mathbb{F}$ be large enough such that there exists a superregular matrix

$$\begin{bmatrix} g_0 & g_1 & \cdots & g_{\ell-1} \end{bmatrix} \in \mathbb{F}^{n \times \ell}$$

and define

$$G(z_1, z_2) = \sum_{0 \leq i+j \leq \delta} G_{ij} z_1^i z_2^j \in \mathbb{F}[z_1, z_2]^n$$

where $G_{ij} = g_{\mu(i,j)}$ and $\mu : \mathbb{N}^2 \to \mathbb{N}^2$ is the map defined by

$$\mu(i,j) = j + \frac{(i+j)(i+j+1)}{2} \text{ for all } (i,j) \in \mathbb{N}^2.$$

Then, if $n \geq \ell$, the 2D convolutional code with encoder $G(z_1, z_2)$ is an ***MDS 2D convolutional code of rate $1/n$ and degree δ****.*

Construction of MDS 3D convolutional codes

Theorem

Let α *be a primitive element of a finite field* $\mathbb{F}_{p^N}$ *with* $N \in \mathbb{N}$ *sufficiently large. Let* $n \geq 6$ *and* $\hat{G}(z_1, z_2) = \sum_{0 \leq a+b \leq 2} G_{ab} z_1^a z_2^b \in \mathbb{F}[z_1, z_2]^{n \times 2}$ *with* $G_{ab} = \left[g_{i,j}^{(a,b)}\right] \in \mathbb{F}^{n \times 2}$ *defined by*

$$g_{i,j}^{(a,b)} = \begin{cases} \alpha^{2^{(3a+b)n+i+j-2}} & \text{if } 0 \leq a+b \leq 1 \\ \alpha^{2^{(3a+b)n+i+j-2}} & \text{if } a+b=2 \text{ and } j \leq 1 \\ 0 & \text{if } a+b=2 \text{ and } j>1 \\ 0 & \text{if } a+b>2. \end{cases}$$

and write

$$\hat{G}(z_1, z_2) = \left[G^{(0)}(z_1, z_2)\ G^{(1)}(z_1, z_2)\right].$$

Define

$$G(z_1, z_2, z_3) = G^{(0)}(z_1, z_2) + G^{(1)}(z_1, z_2) z_3 + G^{(2)}(z_1, z_2) z_3^2,$$

where $G^{(2)}(z_1, z_2) = G_{002} \in \mathbb{F}^n$ *is a vector with all the entries different from zero. Then*

$$\mathcal{C} = \mathrm{Im}_{\mathbb{F}[z_1, z_2, z_3]}\, G(z_1, z_2, z_3)$$

is a 3D MDS convolutional code of rate $1/n$ *and degree* 2.

Construction of MDS mD Convolutional Codes of Rate 1/n

- We develop new constructions of MDS mD convolutional codes based on superregularity of certain matrices.
- Any submatrix of a superregular matrix remains superregular, and row permutations preserve this property.

Lemma

Let $A \in \mathbb{F}_q^{r \times s}$ with $r \leq s$ be a superregular matrix. Then, for any $u \in \mathbb{F}_q^r \setminus \{0\}$, we have that

$$\mathrm{wt}(uA) \geq s - \mathrm{wt}(u) + 1.$$

Construction of MDS mD Convolutional Codes of Rate 1/n

- Superregular matrices were used by Lieb and Pinto [10] to construct optimal 1D MDS convolutional codes.

Given a polynomial matrix $G(z) = \sum_{i=0}^{\delta} G_i z^i \in \mathbb{F}[z]^{1\times n}$, with $G_\delta \neq 0$, let us consider the $(\delta+1)\times n$ constant matrix

$$\Phi_1(G(z)) = \begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_\delta \end{bmatrix}.$$

Theorem ([10])

Let $n, \delta \in \mathbb{N}$ with $n \geq \delta + 1$ and $G(z) = \sum_{i=0}^{\delta} G_i z^i \in \mathbb{F}[z]^{1\times n}$, with $G_\delta \neq 0$. If $\Phi_1(G(z))$ is superregular, then $G(z)$ is a generator matrix of an ***MDS 1D convolutional code of rate*** $1/n$ ***and degree*** δ*.*

- We establish an **upper bound on the free distance of 1D convolutional codes** with **specific row degrees** and also provide the conditions under which these codes achieve that bound.

Theorem

Let $\ell \in \mathbb{N}_0$ and $G_i(z) \in \mathbb{F}_q[z]^{k_i \times n}$ with $k_i \in \mathbb{N}$ for $i = 0, 1, \ldots, \ell - 1$ and $G_\ell(z) \in \mathbb{F}_q[z]^{1\times n}$ such that $G(z) = \begin{bmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_\ell(z) \end{bmatrix} \in \mathbb{F}[z]^{(k_0+\cdots+k_{\ell-1}+1)\times n}$ is the generator matrix of a convolutional code $\mathcal{C}$, where, for $i = 0, 1, \ldots, \ell$, all row degrees of $G_i(z)$ are equal to $\nu + \ell - i$ for some $\nu \in \mathbb{N}$. Then

$$d_{free}(\mathcal{C}) \leq n(\nu + 1).$$

Moreover, let $\mathcal{G} = \begin{bmatrix} \Phi_1(G_0(z)) \\ \Phi_1(G_1(z)) \\ \vdots \\ \Phi_1(G_\ell(z)) \end{bmatrix} \in \mathbb{F}^{L\times n}$ where $L = \sum_{j=0}^{\ell-1} k_j(\nu + \ell + 1 - j) + \nu + 1$. If $n \geq L$ and $\mathcal{G}$ is superregular, then $d_{free}(\mathcal{C}) = n(\nu + 1)$.

Construction of MDS mD Convolutional Codes of Rate 1/n

- Given a polynomial matrix $G(z_1, z_2, \ldots, z_m) = \sum_{i=0}^{\mu} G_i(z_1, z_2, \ldots, z_{m-1}) z_m^i \in R^{1\times n}$, with $G_\mu(z_1, z_2, \ldots, z_{m-1}) \neq 0$, we define the constant matrix for $m \geq 2$:

$$\Phi_m(G(z_1, z_2, \ldots, z_m)) = \begin{bmatrix} \Phi_{m-1}(G_0(z_1, z_2, \ldots, z_{m-1})) \\ \Phi_{m-1}(G_1(z_1, z_2, \ldots, z_{m-1})) \\ \vdots \\ \Phi_{m-1}(G_\mu(z_1, z_2, \ldots, z_{m-1})) \end{bmatrix}$$

Lemma

Let $m, \nu \in \mathbb{N}$. Then

$$\frac{(\nu+m)!}{\nu! m!} = \sum_{i=0}^{\nu} \frac{(\nu+m-i-1)!}{(\nu-i)!(m-1)!}$$

Free Distance Bounds for mD Codes

Theorem

Let $G(z_1, z_2, \ldots, z_m) = \begin{bmatrix} G_0(z_1, z_2, \ldots, z_m) \\ G_1(z_1, z_2, \ldots, z_m) \\ \vdots \\ G_\ell(z_1, z_2, \ldots, z_m) \end{bmatrix} \in R^{(k_0+\cdots+k_{\ell-1}+1)\times n}$ *be a generator matrix of an mD convolutional code* $\mathcal{C}$*, with* $G_0(z_1, \ldots, z_m) \in \mathbb{F}_q[z_1, \ldots, z_m]^{k_0\times n}, \ldots, G_\ell(z_1, \ldots, z_m) \in \mathbb{F}_q[z_1, \ldots, z_m]^{1\times n}$ *of degrees* $\nu_0 \geq \nu_1 \geq \cdots \geq \nu_{\ell-1} > \nu_\ell$*, respectively. Then,*

$$d_{free}(\mathcal{C}) \leq n\frac{(\nu_\ell + m)!}{\nu_\ell!m!}.$$

Moreover, if $n \geq k_0(\nu_0 + 1) + k_1(\nu_1 + 1) + \cdots + k_{\ell-1}(\nu_{\ell-1+1}) + \nu_\ell + 1$ *and* $\Phi_m(G(z_1, z_2, \ldots, z_m))$ *is superregular, then*

$$d_{free}(\mathcal{C}) = n\frac{(\nu_\ell + m)!}{\nu_\ell!m!}.$$

MDS mD Constructions: Rate $1/n$ and Rate k/n

Theorem

*Let $G(z_1, z_2, \ldots, z_m) \in R^{1\times n}$ with row degree equal to δ and $n \geq \delta + 1$, such that $\Phi_m(G(z_1, z_2, \ldots, z_m))$ is superregular. Then $G(z_1, z_2, \ldots, z_m)$ is a generator matrix of an **MDS mD convolutional code of rate** $1/n$ **and degree** δ.*

MDS mD Constructions: Rate $1/n$ and Rate k/n

Theorem

*Let $G(z_1, z_2, \ldots, z_m) \in R^{1\times n}$ with row degree equal to δ and $n \geq \delta + 1$, such that $\Phi_m(G(z_1, z_2, \ldots, z_m))$ is superregular. Then $G(z_1, z_2, \ldots, z_m)$ is a generator matrix of an **MDS mD convolutional code of rate** $1/n$ **and degree** δ.*

Theorem

Let $n \geq \delta + k = k(\nu + 2) - 1$. Let

$$G(z_1, z_2, \ldots, z_m) = \begin{bmatrix} G_0(z_1, z_2, \ldots, z_m) \\ G_1(z_1, z_2, \ldots, z_m) \\ \vdots \\ G_{k-1}(z_1, z_2, \ldots, z_m) \end{bmatrix} \in R^{k\times n}$$

*be a generator matrix of an mD convolutional code $\mathcal{C}$, with $G_i(z_1, z_2, \ldots, z_m)$ of degree $\nu + 1$, $i = 0, 1, \ldots, k-2$, and $G_{k-1}(z_1, z_2, \ldots, z_m)$ of degree ν, such that $\Phi_m(G(z_1, z_2, \ldots, z_m))$ is superregular. Then $G(z_1, z_2, \ldots, z_m)$ is a generator matrix of an **MDS mD convolutional code of rate** k/n **and degree** $\delta = k\nu + k - 1$.*

Conclusion

- We considered mD convolutional codes and we established an upper bound on the distance of these codes.
- We presented concrete constructions of 3D convolutional codes of rate $1/n$ and degree δ that attain such bound for $n \geq 6$ and $\delta \leq 2$.
- We construct new families of MDS mD convolutional codes of rate $1/n$, relying on generator matrices with specific row degree conditions.

Future Research Directions

- Exploring generator matrices with **different row degree conditions**.
- Considering **different superregular structures**.
- Finding **explicit constructions** over small finite fields.

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